Propagation of an electromagnetic pulse in a birefringent material: Waveplates and Retarders

Federico Vismarra[1]

[1] *PhD Candidate @ Attosecond Research Center, Department of Physics, Politecnico of Milano, Italy*

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Introduction

A birefringent medium is characterized by two different refractive indexes among two orthogonal axes. As a consequence of this propriety, this material is used to change electromagnetic fields' polarisation states.

In books, one can typically find exhaustive discussions on how monochromatic waves propagates in such a medium.
Nowadays, the number of scientists using short-light pulses has broadly expanded; thus, more and more young students, including myself, wonder how to appropriately adapt the "old-school discussion" to this more recent framework. Tackling this question is what these few pages are about.

Since I was not finding many online resources that satisfy me, I have decided to write down few equations to properly contextualize the problem and then run few simulations to prove that everything was working as expected.

In the first part, I will introduce the theoretical framework that allows understanding the problem.
In the second part, I will present the simulation result obtained with a simple Matlab code that you can download freely from my GitHub account[1].

[1] GITHUB: fedefirefox

Theoretical discussion

Let us consider a *birefringent linear medium* with slow axis, s, and fast axis, f, orthogonal to the propagation direction, z, which is coincident with the optical axis. In this medium, we let a *plane-wave* electromagnetic field propagates among the optical axis.

The wave equations can be derived from Maxwell equations as:

$$\nabla^2 E_s - \frac{1}{c^2}\frac{\partial^2 E_s}{\partial t^2} = \mu_0 \frac{\partial^2 P_s}{\partial t^2} \tag{1}$$

$$\nabla^2 E_f - \frac{1}{c^2}\frac{\partial^2 E_f}{\partial t^2} = \mu_0 \frac{\partial^2 P_f}{\partial t^2} \tag{2}$$

where c is the speed of light in vacuum, μ_0 is the vacuum permeability, $E_s(\mathbf{r}, t)$ and $E_p(\mathbf{r}, t)$ are the electric field component among the slow and fast axis respectively, and $P_s(\mathbf{r}, t)(P_f(\mathbf{r}, t))$ is the linear polarization among the slow(fast) axis.

Our goal is to find a complete solution for the previous equations. In other words, our task can be summarize as:
Given a medium with a given relation between the slow and fast axis and given a certain input fields, we have to find the pulse space-time evolution at the output.

Propagation of a pulse in a dispersive medium

As a matter of fact, Eqs.(1) and (2) are decoupled so we can solve them independently for generic E and P fields, and unify them later to properly take into account the vectorial nature of the electromagnetic field.

Since we are dealing with plane wave pulses, both the E and P fields are two-dimensional scalar fields, which are functions only of the position among the optical axis, z, and of time, t.
As a consequence, we can re-formulate the wave equations in the Fourier space so that the PDE of Eq.(1), or (2), becomes a simple ODE.

$$\frac{d^2\tilde{E}}{dz^2} + \frac{\omega^2}{c^2}\tilde{E} = -\mu_0\omega^2\epsilon_0\chi^{(1)}\tilde{E} \tag{3}$$

where i) we have exploited the definition of linear polarization in a birefringent material,

$$\tilde{P}(\omega, z) = \epsilon_0\chi^{(1)}(\omega)\tilde{E}(\omega, z)$$

and ii) we have defined the Fourier transform of a generic field, F, as:

$$\tilde{F}(\omega, z) = \int_{-\infty}^{\infty} F(z, t)e^{-i\omega t}dt$$

At this point, let us use the following ansatz for the electric field temporal evolution:

$$E(t, z) = A(t, z)e^{i\omega_c t - ik_c z} \tag{4}$$

$$\tilde{E}(\omega, z) = \tilde{A}(\omega - \omega_c)e^{-ik_c z} \tag{5}$$

where $A(z, t)$ is the so-called complex envelope, ω_c is the central frequency of the pulse and k_c is the central wavevector[2]. If we plug in Eq.(3) the ansatz of Eq.(5) and if we exploit the slowly-varying envelope approximation[3], we achieve:

$$-2ik_c\frac{d\tilde{A}}{dz} + (\frac{\omega^2}{c^2}n(\omega) - k_c^2)\tilde{A} = 0 \tag{6}$$

[2]In Eq. (4) we are using the complex notation for the electric field. Since E-field is a physical observable to make physical sense the real part of Eq. (4) should be taken after any calculation.

[3]This approximation allows us to neglect the second derivative with respect the position, z, of the complex envelope.

where we have defined the refractive index as:

$$n(\omega) = 1 + \chi^{(1)}(\omega)$$

As a last approximation, we assume that $k(\omega) = \frac{\omega^2}{c^2} n(\omega)$ is close enough to k_c for all the relevant frequencies of the pulse.

$$k^2(\omega) - k_c^2 \approx 2k_c(k(\omega) - k_c)$$

As a consequence, Eq. (6) becomes:

$$\frac{d\tilde{A}}{dz} + i(\frac{\omega}{c} n(\omega) - k_c)\tilde{A} = 0 \tag{7}$$

whose solution is simply:

$$\tilde{A}(\omega, z) = \tilde{A}(\omega, 0) e^{-i(\frac{\omega}{c} n(\omega) - k_c)z} \tag{8}$$

Eq. (8) is the formal solution we were looking for. However a little more effort is still needed to tune this result on the initially stated goal.

Pulse propagation in a birefringent medium

We have found formal solution to Eqs. (1) and (2) as:

$$\tilde{A}_s(\omega - \omega_c, z) = \tilde{A}_s(\omega - \omega_c, 0) e^{-i(\frac{\omega}{c} n(\omega) - k_c)z} \tag{9}$$

$$\tilde{A}_f(\omega - \omega_c, z) = \tilde{A}_f(\omega - \omega_c, 0) e^{-i(\frac{\omega}{c} n(\omega) - k_c)z} \tag{10}$$

The Fourier transform of the vectorial field that is present in the medium can be written as:

$$\tilde{\mathbf{E}}(\omega, z) = \tilde{E}_f(\omega, z)\hat{\mathbf{u}}_f + \tilde{E}_s(\omega, z)\hat{\mathbf{u}}_s \tag{11}$$

where $\hat{\mathbf{u}}_s$ and $\hat{\mathbf{u}}_f$ are the unitary vectors of the slow and fast axes, respectively. If we combine the relation of Eq. (5) with the result of Eqs. (9-10) in Eq. (11) we obtain:

$$\tilde{\mathbf{E}}(\omega, z) = \left[\tilde{A}_s(\omega - \omega_c, 0)\hat{\mathbf{u}}_s + \tilde{A}_f(\omega - \omega_c, 0) e^{i\frac{\omega}{c}(n_f(\omega) - n_s(\omega))}\hat{\mathbf{u}}_f\right] e^{-i(\frac{\omega}{c} n_s(\omega) - k_c)z} e^{-ik_c z} \tag{12}$$

It is common to find in literature and in commercial waveplates the *retardation* ϕ which is defined as:

$$\phi = \frac{2\pi}{\lambda}(n_f(\lambda) - n_s(\lambda))L \tag{13}$$

where L is the total length of the material through which the pulse is propagating and λ is the wavelength in vacuum.
The retardation expresses the phase difference accumulated in propagation between the spectral component among fast axis with respect the one at the same frequency among the slow axis.
In our context, it is more useful to define the *retardation per unit length*, δ,

$$\delta = \frac{2\pi}{\lambda}(n_f(\lambda) - n_s(\lambda)) = \frac{\omega}{c}(n_f(\omega) - n_s(\omega))$$

Hence,

$$\boxed{\tilde{\mathbf{E}}(\omega, z) = \left[\tilde{A}_s(\omega - \omega_c, 0)\hat{\mathbf{u}}_s + \tilde{A}_f(\omega - \omega_c, 0) e^{i\delta(\omega)z}\hat{\mathbf{u}}_f\right] e^{-i(\frac{\omega}{c} n_s(\omega) - k_c)z} e^{-ik_c z}} \tag{14}$$

Eq. (14) is already our final solution; however, we can manipulate it a bit further in order to extract some relevant information.

For example, it could be more common to find reported in commercial product the retardation, ϕ, plus some dispersion information, instead of the full refractive index. In that case, it is enough to expand $\frac{\omega}{c}n_s(\omega) - k_c$ in its Taylor series centred at ω_c. As a result:

$$\tilde{\mathbf{E}}(\omega, z) = \left[\tilde{A}_s(\omega-\omega_c, 0)\hat{\mathbf{u}}_s + \tilde{A}_f(\omega-\omega_c, 0)e^{i\delta(\omega)z}\hat{\mathbf{u}}_f\right]e^{-i\frac{\omega-\omega_c}{v_g}z}e^{-iGVD\frac{(\omega-\omega_c)^2}{2}z}e^{-ik_cz} \tag{15}$$

where v_g is the group velocity, GVD is the group velocity dispersion [4].

In practical situations, the laboratory reference system, which is used to describe the electric field polarization, differs from the slow and fast axes. In other terms, one has the electric field defined on the x- and y- axes set, both orthogonal to the optical axis z, and can vary the angle between this reference system and the birefringent material slow and fast axes.

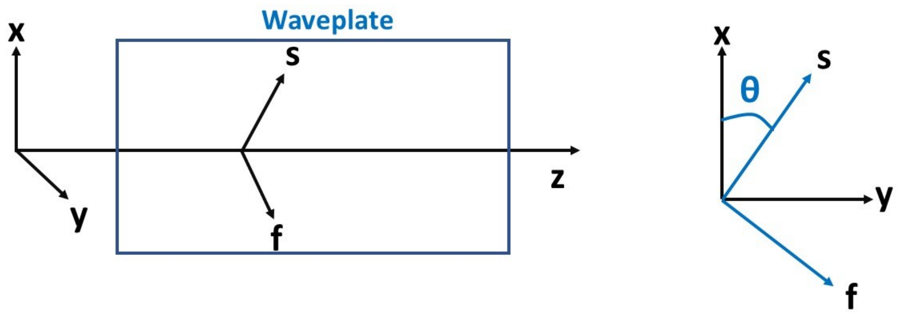

Figure 1: *Representation of the laboratory reference system (x-y-z) vs the birefringent material reference system (s-f-z).*

The trasformation rule between the electric field in the laboratory reference system and the birefringet material one is given by:

$$\begin{bmatrix} E_s \\ E_f \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \tag{16}$$

where θ, as shown in Figure 1, is the angle between the x-axis and the s-axis.
By reversing the above transformation matrix, one obtains:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} E_s \\ E_f \end{bmatrix} \tag{17}$$

Therefore, we may express Eq. (14) in the laboratory reference system as:

$$\boxed{\tilde{\mathbf{E}}(\omega, z) = \left[(\cos\theta\tilde{A}_s(0) - \sin\theta\tilde{A}_f(0)e^{i\delta(\omega)z})\hat{\mathbf{u}}_x + (\sin\theta\tilde{A}_s(0) + \cos\theta\tilde{A}_f(0)e^{i\delta(\omega)z})\hat{\mathbf{u}}_y\right]e^{-i(\frac{\omega}{c}n_s(\omega)-k_c)z}e^{-ik_cz}}$$

Generally speaking, given a material of length L:

- If $\phi = \delta L = n\pi$, with n non-null integer, the material is called *half-waveplate*, cause L is a multiple of $\lambda/2$.
- If $\phi = (2n+1)\frac{\pi}{2}$, with n integer, the material is called *quarter-waveplate*, cause L is an odd multiple of $\lambda/4$.
- If $\phi = A(\omega - \omega_c)$, with A a constant, the material is a pure-retarder or simply a birefringent thick plate.

The width of the spectrum, ω, for which one of the above conditions is satisfied, defines the spectral range in which the material can effectively operate.

[4]Usually one finds the GDD, group delay dispersion, which is equal to GVD times the length of the material. Moreover, it is not common to find v_g, thus, in most calculation it can be put equal to c.

Simulations results

In the following, few simulations that exploit the result of Eq. (14) are performed for different values of the retardation[5]. As shown in Figure 2, in the simulations, we use a 5 fs pulse[6], centred at 800 nm.

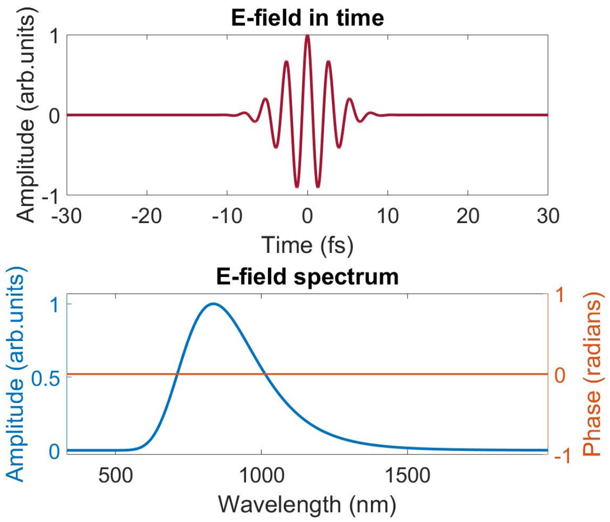

Figure 2: *Temporal and spectral representation of the electromagnetic pulse used in simulations.*

We will assume a medium thin enough, $20\ \mu m$, to neglect the dispersion effects discussed in Eq. (15). In all simulations, the slow axis is tilted of $\theta = 45^o$ with respect the x-axis, in this way the "ideal" behavior of the waveplates will be tested.

In Figure 3, a x-polarized beam is transformed into a y-polarized one after propagation in a *half-waveplate*. The waveplate satisfies the request $\phi = \delta L = \pi$ for all the frequency (flat response). Since the angle of incidence is 45^o, the x-component of the electric field is equally split between the s-axis and the f-axis. Progressively, the waveplate provides a phase-mismatch between the two components, resulting in a 90-degrees change of polarization at the crystal exit.

In Figure 4, a x-polarized pulse is completely turned into a left-hand circular polarization (anti-clocked-wise from the positive z-axis looking toward the negative values). This fact happens because the linear polarized pulse enters into a quarter-waveplate with retardance $\phi = \delta L = \pi$ for all frequency.

In Figure 5, we are discussing a slightly different process where the retardance is engineered to be a linear function around the central frequency of the pulse. This object is called a *retarder*. Such plate, when used at 45^o, allows to split the pulse in two sub-pulses among the s- and f- axes, each propagating at a different group velocity. As a result, the two replicas gain in propagation a fixed temporal delay, $\Delta\tau$.

$$\phi = \left(\frac{L}{v_{gf}} - \frac{L}{v_{gs}}\right)(\omega - \omega_c)$$
$$\phi = \Delta\tau(\omega - \omega_c)$$

If the relative delay between the two components is below the duration of the pulses, a region in space with a mixed polarization will exist, as shown in Figure 5.

[5] The code is available at my GitHub username: fedefirefox
[6] duration in intensity

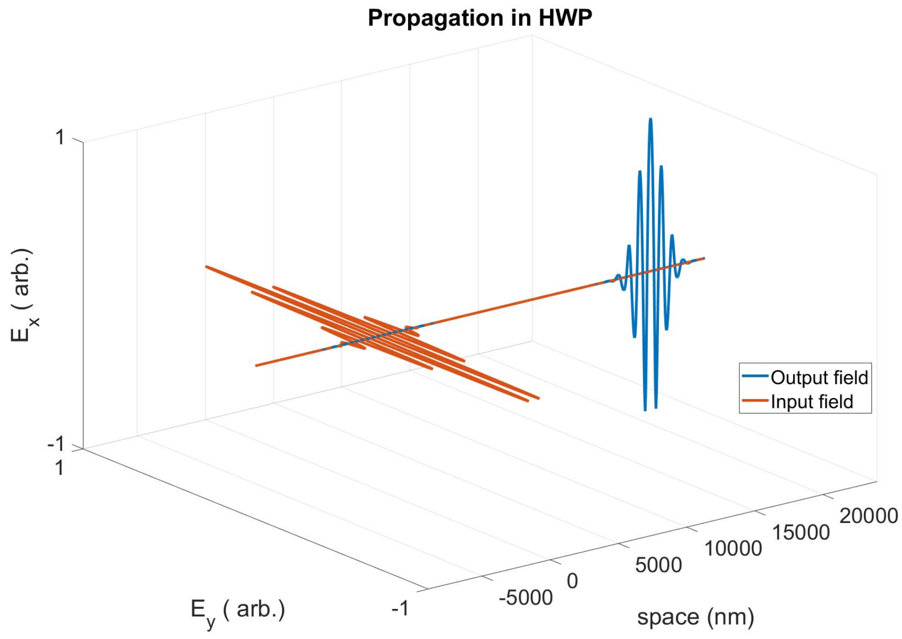

Figure 3: *Change of polarization in a half-waveplate. From a x-polarized field, the birefringent material is capable of switching the polarization to the y-axis. In the assumption of dispersionless material, this effect works without altering the pulse duration.*

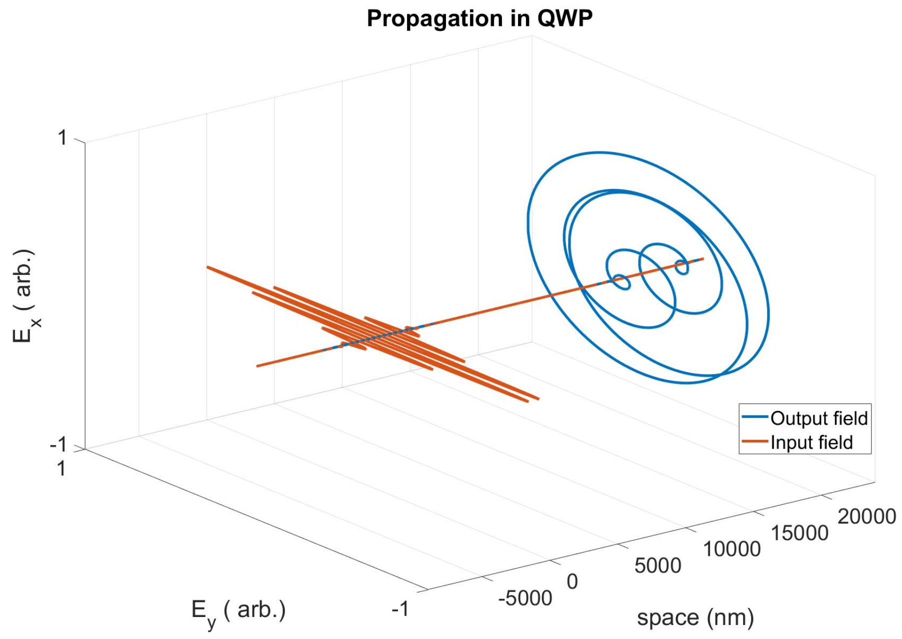

Figure 4: *Change of polarization in a quarter-waveplate. The birefringent material is capable of switching the polarization from a x-polarized field to a left-hand polarization (or right-hand polarization if we rotate the θ angle of 90 degrees with respect the actual value of 45^o).*

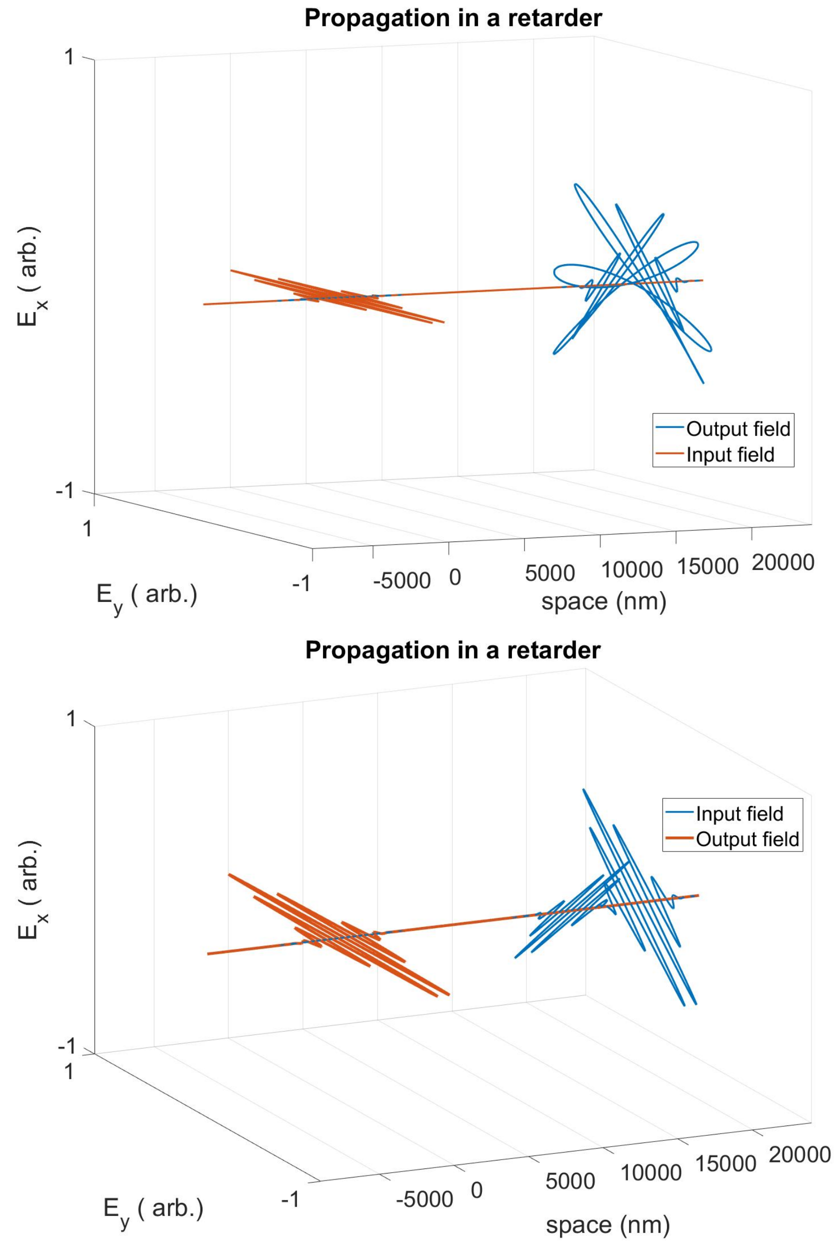

Figure 5: *In the top panel, the delay between the two replicas is less than the pulse duration. Consequently, there is a region where the two polarizations mix resulting in a non-trivial time-dependent ellipticity. In the lower panel, instead, the delay between the two replicas is greater than the pulse duration. Hence, the two sub-pulses are distinguishable in space-time and do not mix.*

Conclusion

Many studies and models can arise from this simple discussion.
For example, one can investigate the effect of multiples waveplates to create a pulse with a custom shape. This is the case of the polarization gating technique used to generate isolated attosecond pulse. In this technique, a combination of a thick plate set a 45^o, and a quarter-waveplate is used to create a pulse that presents a single-cycle linearly polarized electric field preceded and followed by circular polarization.
One can also discuss the effect of frequency-dependent retardance on the waveplates performances. This is the case of real-life retardance. One can model their behaviour by assuming the frequency response of an ideal half-waveplate times a gaussian centred at the central pulse frequency or by simply

downloading the real frequency response from the product datasheet.

This small work is far from being conclusive nor particularly interesting per se. However, at this point, it would be just a matter of exploring the "phase space" of parameters one can vary[7] and discuss the different outcomes in terms of polarization states and some other figures of merit one can find in literature or can invent. Since I have to move on with other tasks, I must leave this funny game to the reader.

[7] Input polarization, dispersion of the pulse, dispersion of the medium, non-flat frequency response, combinations of waveplates, etc.